

# Creating multi-photon polarization bound-entangled states

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## Abstract

Bound entangled states are the exotic objects in the entangled world. They require entanglement to create them, but once they are formed, it is not possible to locally distill any free entanglement from them. It is only until recently that a few bound entangled states were realized in the laboratory. Motivated by these experiments, we propose schemes for creating various classes of bound entangled states with photon polarization. These include Acín-Bruß-Lewenstein-Sanpara states, Dür's states, Lee-Lee-Kim bound entangled states, and an unextendible-product-basis bound entangled state.

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## I. INTRODUCTION

Entanglement, arguably, one of the most important ingredients in quantum information, has been extensively explored and investigated in recent years [1]. It can exist in the form of pure entangled states such as Bell states, which enable many quantum information processing protocols [2]. In the regime of mixed states, entanglement can exhibit more features [1]. Of these, the bound entangled states [3] have very fragile entanglement properties; they require entanglement to create them, but once they are formed, there is no way of locally distilling any useful free entanglement from them. Bound entangled states lie on the border of entangled states with un-entangled states. While the characteristics of bound entanglement are interesting theoretically in their own right, they are usually considered to be of little practical use, analogous to heat in thermodynamics. Nevertheless, some bound entangled states have found application in information concentration [4], bi-partite activation [5], multi-partite superactivation [6] and secure key distillation [7, 8], as well as providing a resource for certain zero-capacity quantum channels [9]. It is likely that more applications with bound entanglement will be found. Nevertheless, it was not until very recently that the synthesis of certain bound entangled states was attempted in the laboratory, including multi-qubit states [10–13] and continuous-variable states [14].

This paper considers a collection of multi-qubit bound entangled states of various types, including Acín-Bruß-Lewenstein-Sanpera three-qubit states [15], Smolin’s four-qubit state [16], Dür’s  $N$ -qubit states [17], Lee-Lee-Kim  $N$ -qubit bound entangled states [18] and an unextendible-product-basis (UPB) bound entangled state [19]. Motivated by recent experiments on creating bound entangled states [10–14], we propose schemes for creating these states using photon polarization.

## II. SMOLIN’S BOUND ENTANGLED STATE

This section gives a brief review of the ideas in Refs. [11, 12]; see also Ref. [20]. The key point is that mixed states are created by a statistical mixture of pure states.

The Smolin state [16] is a four-qubit mixed state

$$\rho_{\text{Smolin}}^{ABCD} \equiv \frac{1}{4} \sum_{i=0}^3 (|\Psi_i\rangle\langle\Psi_i|)_{AB} \otimes (|\Psi_i\rangle\langle\Psi_i|)_{CD}, \quad (1)$$

where the  $|\Psi\rangle$ 's are the four Bell states  $|\Phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$  and  $|\Psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$ . The form (1) suggests that the Smolin state can be generated by mixing the four terms of the four-particle states, each being a product of two Bell pairs. Note that  $|\Phi^-\rangle = \mathbb{1} \otimes \sigma_z |\Phi^+\rangle$ ,  $|\Psi^+\rangle = \mathbb{1} \otimes \sigma_x |\Phi^+\rangle$ , and  $|\Psi^-\rangle = -i\mathbb{1} \otimes \sigma_y |\Phi^+\rangle$  (where  $\mathbb{1}$  and  $\sigma$ 's stand for the identity and the three Pauli operators, respectively); namely, the four Bell states can be locally converted into one another. Once two pairs of  $|\Phi^+\rangle$  states have been produced using, e.g., downconversion [21], one only needs to randomly and simultaneously apply either one of  $\mathbb{1}$ ,  $\sigma_z$ ,  $\sigma_x$  and  $\sigma_y$  with equal probability to one photon of each entangled pair. The resultant statistical mixture will be the Smolin state, as was done in Refs. [11, 12]

We remark that the entanglement in the Smolin state can be unlocked if two of the parties can perform a joint Bell-state analysis [16]. This state can be used, e.g., in information concentration [4] and multipartite superactivation [6]. As far as the amount of entanglement is concerned, for the Smolin state the negativity  $\mathcal{N}$  [1, 22] is zero for any 2:2 partitioning, e.g.,  $\{\text{AB} : \text{CD}\}$ , but nonzero for 1:3 partitioning, e.g.,  $\{\text{A}:\text{BCD}\}$ . Specifically,  $\mathcal{N}_{\text{A}:\text{BCD}} = 1$  but  $\mathcal{N}_{\text{AB}:\text{CD}} = 0$ . Furthermore, the Smolin state has the same amount of entanglement as the Greenberger-Horne-Zeilinger-(GHZ) state, as quantified by the geometric measure of entanglement [23, 24].

Note that the state  $\rho_{\text{Smolin}}^{ABCD}$  can also be written as

$$\rho_{\text{Smolin}}^{ABCD} = \frac{1}{4} \sum_{i=0}^3 |X_i\rangle\langle X_i|, \quad (2)$$

where the  $|X\rangle$ 's are the four orthogonal GHZ states:

$$\begin{aligned} |X_0\rangle &\equiv \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle), & |X_1\rangle &\equiv \frac{1}{\sqrt{2}}(|0011\rangle + |1100\rangle), \\ |X_2\rangle &\equiv \frac{1}{\sqrt{2}}(|0101\rangle + |1010\rangle), & |X_3\rangle &\equiv \frac{1}{\sqrt{2}}(|0110\rangle + |1001\rangle). \end{aligned}$$

This also provides an alternative way of creating the Smolin state using a four-photon GHZ state as a resource. In practice, the four-photon GHZ state is also created from two pairs of Bell states [25]. For many other states below it is essential to use a GHZ state as a resource state. We remark that Barreiro et al. synthesized this bound entangled state with trapped ions, first by creating an appropriate diagonal state, followed by a four-qubit unitary entangling gate [13].

### III. ACÍN-BRUSS-LEWENSTEIN-SANPERA BOUND ENTANGLED STATES

Acín, Bruß, Lewenstein and Sanpera (ABLS) have proposed a class of three-qubit bound entangled states [15], described by

$$\rho_{\text{ABLS}}(a, b, c) = \frac{1}{n} \left( 2|3\text{GHZ}\rangle\langle 3\text{GHZ}| + c|001\rangle\langle 001| + \frac{1}{c}|110\rangle\langle 110| + b|010\rangle\langle 010| + \frac{1}{b}|101\rangle\langle 101| + a|100\rangle\langle 100| + \frac{1}{a}|011\rangle\langle 011| \right), \quad (3)$$

where  $|3\text{GHZ}\rangle$  is the three-qubit GHZ state  $|3\text{GHZ}\rangle \equiv (|000\rangle + |111\rangle)/\sqrt{2}$  and the parameter  $n$  is a normalization factor  $n \equiv 2 + a + 1/a + b + 1/b + c + 1/c$ , and parameters  $a, b$  and  $c$  satisfy  $a, b, c > 0$  and  $abc \neq 1$ . The last condition, derived using the so-called range criterion [1], ensures that the ABLS state is entangled [15]. This family of states can be rewritten as

$$\rho_{\text{ABLS}}(a, b, c) = \frac{1}{n} \left( 2|3\text{GHZ}\rangle\langle 3\text{GHZ}| + \rho_a^{(+)} + \rho_a^{(-)} + \rho_b^{(+)} + \rho_b^{(-)} + \rho_c^{(+)} + \rho_c^{(-)} \right), \quad (4)$$

where the un-normalized states  $\rho_{a,b,c}^{(\pm)} \equiv |\psi_{a,b,c}^{(\pm)}\rangle\langle\psi_{a,b,c}^{(\pm)}|$ 's are defined via the following un-normalized states,

$$|\psi_c^{(\pm)}\rangle \equiv \sqrt{\frac{c}{2}}|001\rangle \pm \frac{1}{\sqrt{2c}}|110\rangle \quad (5)$$

$$|\psi_b^{(\pm)}\rangle \equiv \sqrt{\frac{b}{2}}|010\rangle \pm \frac{1}{\sqrt{2b}}|101\rangle \quad (6)$$

$$|\psi_a^{(\pm)}\rangle \equiv \sqrt{\frac{a}{2}}|100\rangle \pm \frac{1}{\sqrt{2a}}|011\rangle. \quad (7)$$

In addition to using the range criteria [1, 15], the existence of entanglement in this family of states can also be detected by entanglement witnesses [26]. That the states are positive under partial transpose (PPT) ascertains that they are undistillable with respect to any bipartition [15]. Being both entangled and undistillable, the above family of states are therefore bound entangled. Kampermann et al. used a liquid-NMR system to implement these states, and they referred to the resultant states as pseudo bound-entangled states as the true states created are mixture of a small relative amount of these bound entangled states with a large amount of the completely mixed state. Strictly speaking, no true entanglement is present in such a system, unless the temperature is very low. It is thus interesting to see whether these bound entangled states can be created in other systems, where genuine

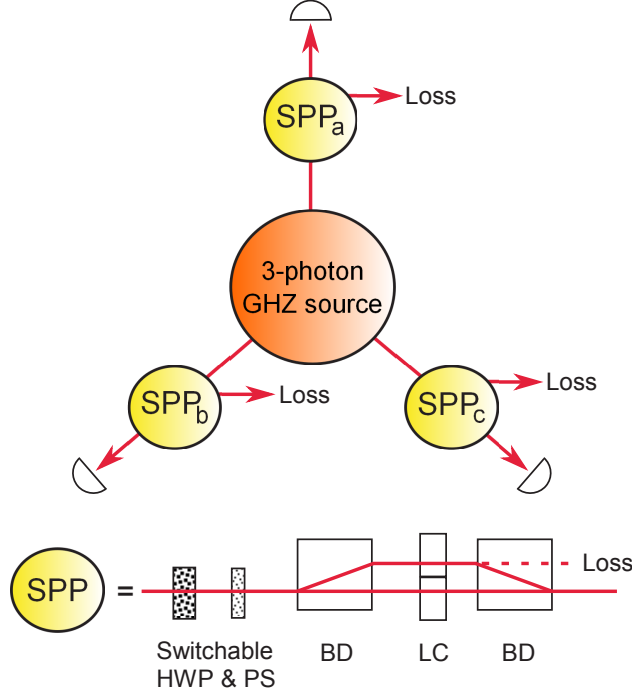


FIG. 1: (color online) Scheme for creating ABLS bound entangled states from a 3-photon GHZ state. *SPP*'s indicate collectively either (1) identity or (2) unitary gates ( $\sigma_x$  or  $\sigma_x\sigma_z$ ) implemented by waveplates, followed by switchable partial polarizers. The indicated loss is due to partial polarizers. HWP stands for the half-wave plate and PS stands for the phase shifter that turns  $V \rightarrow -V$ ; both elements can be switched on and off. BD stands for the beam-displacer (see, e.g., Refs [27, 28]), which separates, say,  $H$  and  $V$  polarizations, and LC stands for the liquid crystal, which acts as a waveplate. The second BD is placed upside down so as to combine the displaced beams. The LCs control the degree of partial polarization, and their action can be switched on and off.

entanglement can be easily achieved. Although ABLS states were only implemented in liquid NMR, other states, such as the Smolin's state [16], have been created in other systems, such as photons [11, 12] and trapped ions [13]. The goal of this section is to provide a scheme for creating ABLS bound entangled states with photon polarization.

Kamperman et al. synthesized their bound entangled state by first creating an appropriate diagonal state, followed by a three-qubit unitary gate [10]. This approach is very difficult with photons, as entangling gates are hard to come by. So how can one create ABLS bound entangled states using photon polarization states? Suppose we have a GHZ state  $(|HHH\rangle + |VVV\rangle)/\sqrt{2}$  (where  $H$  stands for a horizontally polarized photon, and  $V$

is a vertically polarized photon), and, for example, we have a switchable unitary gate (implementable by Pockels cell [27] or liquid crystals [12]) at the path of photon 3 to control the possible actions: (i)  $H \rightarrow V$  and  $V \rightarrow H$ ; (ii)  $H \rightarrow V$  and  $V \rightarrow -H$ ; (iii) do nothing, then the resultant state is (i)  $(|HHV\rangle + |VVH\rangle)/\sqrt{2}$ ; (ii)  $(|HHV\rangle - |VVH\rangle)/\sqrt{2}$ ; or (iii)  $(|HHH\rangle + |VVV\rangle)/\sqrt{2}$ , respectively. Furthermore, if we have a switchable partial polarizer [28] (which is switched on only in the former two cases) acting on photon 3, with polarization-dependent transmissions being  $T_H^{(c)}/T_V^{(c)}$ , then the state  $(|HHV\rangle \pm |VVH\rangle)/\sqrt{2}$  is transformed (probabilistically) to  $\sqrt{T_V^{(c)}/2}|HHV\rangle \pm \sqrt{T_H^{(c)}/2}|VVH\rangle$  (un-normalized). This is the key step to create the  $|\psi_c^{(\pm)}\rangle$  state. Note that  $|\psi_c^{(+)}\rangle$  and  $|\psi_c^{(-)}\rangle$  will be created with equal probability and that one should choose  $T_V^{(c)}/T_H^{(c)} = c^2$ . Such partial polarizers are an important ingredient in implementing general local filtering operations [28, 29], as was used in various places, such as the Procrustean method for entanglement distillation [28, 30] and the construction of optimal witnesses [31]. Similarly, we can have such sets of devices (i.e., switchable waveplates and partial polarizers) placed in the path of the other two photons with transmission coefficients  $T_{H/V}^{(a)}$  and  $T_{H/V}^{(b)}$  (which satisfy  $T_V^{(a)}/T_H^{(a)} = a^2$  and  $T_V^{(b)}/T_H^{(b)} = b^2$ ), respectively, to create  $|\psi_a^{(\pm)}\rangle$  and  $|\psi_b^{(\pm)}\rangle$ . By appropriately mixing these states together with the GHZ state, i.e., firing the three Pockels cells probabilistically to match the relative weight in the state  $\rho(a, b, c)$ , we will create the state at the collection output ports, conditioned on the occurrence of a three-fold coincidence. See Fig. 1 for the schematic setup. In particular, the probabilities that no Pockels cell fires and that either  $a$ -,  $b$ -, or  $c$ -th Pockels cell fires are  $p_{\text{GHZ}}$ ,  $p_a$ ,  $p_b$ , or  $p_c$ , respectively, and they should satisfy

$$p_{\text{GHZ}} : p_a T_V^{(a)} : p_b T_V^{(b)} : p_c T_V^{(c)} = 2 : a : b : c. \quad (8)$$

#### IV. DÜR-CIRAC STATES AND DERIVED BOUND ENTANGLED STATES

It was shown by Dür and Cirac [32] that an arbitrary  $N$ -qubit state  $\rho$  can be locally depolarized into the form

$$\begin{aligned} \rho_{\text{DC}} = & \lambda_0^+ |\Psi_0^+\rangle \langle \Psi_0^+| + \lambda_0^- |\Psi_0^-\rangle \langle \Psi_0^-| \\ & + \sum_{j=1}^{2^{N-1}-1} \lambda_j (|\Psi_j^+\rangle \langle \Psi_j^+| + |\Psi_j^-\rangle \langle \Psi_j^-|), \end{aligned} \quad (9)$$

while preserving  $\lambda_0^\pm = \langle \Psi_0^\pm | \rho | \Psi_0^\pm \rangle$  and  $\lambda_j = \langle \Psi_j^+ | \rho | \Psi_j^+ \rangle + \langle \Psi_j^- | \rho | \Psi_j^- \rangle$ , where  $|\Psi_0^\pm\rangle \equiv (|0^{\otimes N}\rangle \pm |1^{\otimes N}\rangle)/\sqrt{2}$ , and the  $|\Psi_j^\pm\rangle$ 's are GHZ-like states

$$|\Psi_j^\pm\rangle \equiv (|j, 0\rangle \pm |\underline{2^{N-1} - j - 1}, 1\rangle)/\sqrt{2} = (|j_1 j_2 \dots j_{N-1} 0\rangle \pm |\bar{j}_1 \bar{j}_2 \dots \bar{j}_{N-1} 1\rangle)/\sqrt{2}, \quad (10)$$

where  $j = 1, \dots, 2^{N-1} - 1$ ,  $j_1 j_2 \dots j_{N-1}$  is the binary representation of  $j$  with  $j_k = 0$  or  $1$ , and  $\bar{j}_k \equiv 1 - j_k$ . Normalization gives the condition

$$\lambda_0^+ + \lambda_0^- + 2 \sum_{j \neq 0} \lambda_j = 1. \quad (11)$$

Now define  $\Delta \equiv \lambda_0^+ - \lambda_0^-$ , which we assume to be non-negative without loss of generality. Consider a bipartite partitioning  $I_j$  ( $j \neq 0$ ) which divides  $1, 2, \dots, N$  into two groups, with one of them containing indices  $k$  such that  $j_k = 1$  in the binary representation of  $j = j_1 j_2 \dots j_{N-1}$ . The rest of the indices  $m$  (with  $j_m = 0$ ), in addition to  $N$ , are contained in the other group. One can compute the negativity with respect to the partitioning  $I_j$ , and obtains [32]

$$\mathcal{N}_{I_j} = \max\{0, \Delta - 2\lambda_j\}. \quad (12)$$

A sufficient condition to infer that the state is entangled is  $\mathcal{N}_{I_j} > 0$  for certain  $j$ , and this means that  $\Delta > 2\lambda_j$ . On the other hand, when  $\mathcal{N}_{I_j} = 0$ , i.e.,

$$2\lambda_j \geq \Delta,$$

this condition implies that there cannot exist distillable entanglement across the bipartition  $I_j$  [32].

From the form of the states (9), it is easy to see that all the above Dür-Cirac states can be created by the method of mixing, once an  $N$ -partite GHZ state can be generated. In the following we shall discuss Dür and Lee-Lee-Kim bound entangled states and their generalization. All these belong to the class of Dür-Cirac states.

### A. Dür's bound entangled states

Dür [17] found that for  $N \geq 4$  the following state is bound entangled:

$$\rho_D \equiv \frac{1}{N+1} \left( |\Psi_G\rangle \langle \Psi_G| + \frac{1}{2} \sum_{k=1}^N (P_k + \bar{P}_k) \right), \quad (13)$$

where  $|\Psi_G\rangle \equiv (|0^{\otimes N}\rangle + |1^{\otimes N}\rangle)/\sqrt{2}$  is an  $N$ -partite GHZ state;  $P_k \equiv |u_k\rangle\langle u_k|$  is a projector onto the state  $|u_k\rangle \equiv |0\rangle_1|0\rangle_2\cdots|1\rangle_k\cdots|0\rangle_N$ ; and  $\bar{P}_k \equiv |v_k\rangle\langle v_k|$  projects on to  $|v_k\rangle \equiv |1\rangle_1|1\rangle_2\cdots|0\rangle_k\cdots|1\rangle_N$ . It has been shown that this state violates the Mermin-Klyshko-Bell inequality for  $N \geq 8$  [17], and that it violates a three-setting Bell inequality for  $N \geq 7$  [33] and a functional Bell inequality for  $N \geq 6$  [34]. From the experimental point of view, it is better to have a range of parameters that the bound entangled states reside in, as this results in less stringent requirements on the experimental errors [35]. Indeed, it was shown in Ref. [23] that for  $N \geq 4$  the family of states

$$\rho_D(x) \equiv x|\Psi_G\rangle\langle\Psi_G| + \frac{1-x}{2N} \sum_{k=1}^N (P_k + \bar{P}_k), \quad (14)$$

are bound entangled if  $0 < x \leq 1/(N+1)$  and is still entangled but not bound entangled for  $x > 1/(N+1)$ . This can be seen from the fact that the negativities of  $\rho_D(x)$  with respect to the two different partitions  $(1:2\cdots N)$  and  $(12:3\cdots N)$  are

$$\mathcal{N}_{1:2\cdots N}(\rho_D(x)) = \max\{0, [(N+1)x - 1]/N\}, \quad (15a)$$

$$\mathcal{N}_{12:3\cdots N}(\rho_D(x)) = x. \quad (15b)$$

In contrast to Smolin's state, the four-qubit Dür's states have zero negativity with respect to a 1:234 partition but non-zero negativity with respect to a 12:34 partition.

Instead of the original form by Dür [17], we can rewrite  $P_k + \bar{P}_k$  as a mixture of GHZ-like states as follows,

$$P_k + \bar{P}_k = |G_k^+\rangle\langle G_k^+| + |G_k^-\rangle\langle G_k^-|, \quad (16)$$

where

$$|G_k^\pm\rangle \equiv \frac{1}{\sqrt{2}}(|u_k\rangle \pm |v_k\rangle) = \sigma_k^x(\sigma_k^z)^{0/1}|\Psi_G\rangle, \quad (17)$$

where 0 (1) in the exponent of  $\sigma_k^z$  corresponds to  $+$  ( $-$ ). Therefore, we can also rewrite  $\rho_D(x)$  as

$$\rho_D(x) = x|\Psi_G\rangle\langle\Psi_G| + \frac{1-x}{2N} \sum_{k=1}^N \sum_{\alpha=\pm} |G_k^\alpha\rangle\langle G_k^\alpha|, \quad (18)$$

which suggests a way to create this state by mixing up various GHZ-like states, with the probabilities being the corresponding coefficients. See Fig. 2 for the schematic setup.

Dür's bound entangled states and their generalization belong to the general Dür-Cirac states. In particular, in terms of Eq. (9) Dür's state has  $\lambda_0^+ = 1/(N+1)$ ,  $\lambda_0^- = 0$ , and



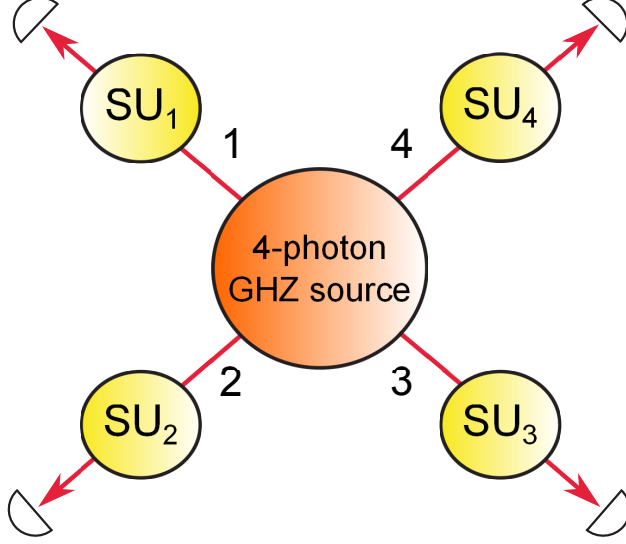


FIG. 2: (color online) Scheme for creating Dür-Cirac states, Dür bound entangled states, LLK bound entangled states. It is illustrated with a 4-photon GHZ state source.  $SU$ 's represent unitary gates that allow to switch between  $\mathbb{1}$ ,  $\sigma_x$  and  $\sigma_x\sigma_z$ .

$\lambda_j = 1/2(N + 1)$  for  $j = 2^0, 2^1, \dots, 2^{N-1}$ , and zero otherwise. We remark that photonic GHZ states of three [36], four [25], five [37] and six photons [38] have all been created with high fidelity, and thus it is within the reach of current technology to implement (generalized) Dür states, as well as all other Dür-Cirac states, with up to six photons.

### B. Lee-Lee-Kim bound entangled states

Lee, Lee and Kim (LLK) [18] constructed bound entangled states that are analogous to Dür's states (but with a different set of  $\lambda$ 's)

$$\begin{aligned} \rho_{\text{LLK}} = & \lambda_0^+ |\Psi_0^+\rangle\langle\Psi_0^+| + \lambda_0^- |\Psi_0^-\rangle\langle\Psi_0^-| \\ & + \sum_{j=1}^{2^{N-1}-1} \lambda_j (|\Psi_j^+\rangle\langle\Psi_j^+| + |\Psi_j^-\rangle\langle\Psi_j^-|), \end{aligned} \quad (19)$$

with  $\lambda_0^+ = 1/(N - 1)$ ,  $\lambda_0^- = 0$ , and  $\lambda_j = 1/2(N - 1)$  if  $j \in J_N \equiv \{3, 6, \dots, 3 \times 2^{N-3}\}$  and  $\lambda_j = 0$  otherwise.

It is easy to see that the LLK states are entangled as they have the negativity (12)  $\mathcal{N} = 2/(N - 1)$  with respect to the partition  $\{1, 2, \dots, N - 1 : N\}$ . It turns out that the non-distillability conditions (13) covered by the various partitionings induced by  $j \in J_N$

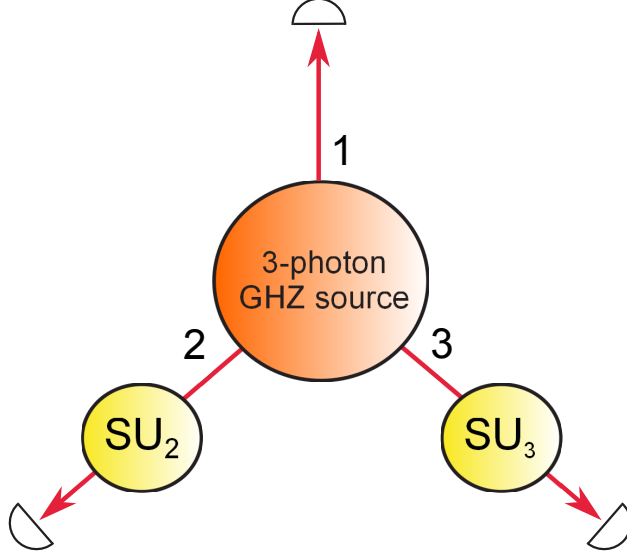


FIG. 3: (color online) Scheme for creating the Chi et al. three-qubit bound entangled state from a 3-photon GHZ state.  $SU$ 's represent a unitary gates, switchable between  $\mathbb{1}$ ,  $\sigma_x$  and  $\sigma_x\sigma_z$ .

are sufficient to conclude that the states are non-distillable and hence bound entangled [18]. Lee, Lee and Kim also showed that for  $N \geq 6$ , the state violates the Mermin-Klyshko-Bell inequality [18].

Similar to Dür's states, we generalize the parameter range of the LLK states:

$$\rho_{\text{LLK}}(x) \equiv x|\Psi_0^+\rangle\langle\Psi_0^+| + \frac{1-x}{2(N-2)} \sum_{j \in j_N} (|\Psi_j^+\rangle\langle\Psi_j^+| + |\Psi_j^-\rangle\langle\Psi_j^-|). \quad (20)$$

They are bound entangled for  $0 < x \leq 1/(N-1)$ .

Chi et al. later showed that for a sufficiently large number  $M$  of the settings in measurement, the LLK bound entangled states violate an  $M$ -setting Bell inequality if and only if  $N \geq 4$  [39]. For  $N = 3$  they instead constructed a simple three-qubit bound entangled state [39]

$$\rho_3 = \frac{1}{3}|\Psi_0^+\rangle\langle\Psi_0^+| + \frac{1}{6} \sum_{j=1,3} (|\Psi_j^+\rangle\langle\Psi_j^+| + |\Psi_j^-\rangle\langle\Psi_j^-|), \quad (21)$$

which violates not a Bell inequality but a positive partial transpose inequality, i.e.,  $|\text{tr}\rho PT_N| \leq 1$  with  $PT_N \equiv 2^{N-1}(|\Psi_0^+\rangle\langle\Psi_0^+| - |\Psi_0^-\rangle\langle\Psi_0^-|)$  [40]. For this state,  $|\text{tr}\rho_3 PT_N| = 4/3$ .

This state can be generalized to

$$\rho_3(x) = x|\Psi_0^+\rangle\langle\Psi_0^+| + \frac{1-x}{4} \sum_{j=1,3} (|\Psi_j^+\rangle\langle\Psi_j^+| + |\Psi_j^-\rangle\langle\Psi_j^-|), \quad (22)$$

such that it is still bound entangled for  $0 < x \leq 1/3$ . Due to the small number of qubits and the small number in the constituent pure states, this particular state is very easy to create; see Fig. 3 for a schematic setup.

## V. A BOUND ENTANGLED STATE FROM AN UNEXTENDIBLE PRODUCT BASIS

Our example of an unextendible product basis (UPB) involves three qubits, A, B, and C:  $|\psi_1\rangle \equiv |0, 0, 0\rangle$ ,  $|\psi_2\rangle \equiv |1, +, -\rangle$ ,  $|\psi_3\rangle \equiv |-, 1, +\rangle$ , and  $|\psi_4\rangle \equiv |+, -, 1\rangle$ , where  $|\pm\rangle \equiv (|0\rangle \pm |1\rangle)/\sqrt{2}$ . A simple analysis shows that there does not exist another linearly-independent *product* state that is orthogonal to all four states  $|\psi_i\rangle$ . Therefore, any state that is orthogonal to the four product basis states must be entangled, and hence the basis is named UPB. Note that the latter three states  $|\psi_2\rangle$ ,  $|\psi_3\rangle$  and  $|\psi_4\rangle$  are related via a periodic shift of all three parties  $A \rightarrow B \rightarrow C \rightarrow A$ , and hence the basis is also called the SHIFTS UPB. From the above properties of UPB [19], Bennett et al. showed that the following three-qubit mixed state is a bound entangled state:

$$\rho_{\text{UPB}} = \frac{1}{4}(\mathbb{1} - \sum_{i=1}^4 |\psi_i\rangle\langle\psi_i|). \quad (23)$$

This state is entangled by construction, as there cannot exist product states in the subspace orthogonal to the UPB basis. Hence, the state  $\rho_{\text{UPB}}$  cannot be expressed in terms of a mixture of product states [19]. Bennett et al. also showed that the UPB bound entangled state has the property of being two-way PPT and two-way separable, i.e., the entanglement across any split into two parties is zero [19]. This can be understood easily, as described in the following. First, the state is invariant under the SHIFT operation. Second, there is one specific decomposition of  $\rho_{\text{UPB}}$  into a mixture of four states which are manifestly two-way

separable (and thus PPT),

$$\rho_{\text{UPB}} = \frac{1}{4} \sum_{i=1}^4 |\phi_i\rangle\langle\phi_i| \quad (24)$$

$$|\phi_1\rangle \equiv \frac{1}{\sqrt{3}}(|01\rangle - |10\rangle + |11\rangle)|0\rangle = |\psi_1\rangle|0\rangle \quad (25)$$

$$|\phi_2\rangle \equiv \frac{1}{\sqrt{12}}(3|00\rangle + |01\rangle - |10\rangle + |11\rangle)|1\rangle = |\psi_2\rangle|1\rangle \quad (26)$$

$$|\phi_3\rangle \equiv \frac{1}{\sqrt{6}}(|01\rangle + 2|10\rangle + |11\rangle)|+\rangle = |\psi_3\rangle|+\rangle \quad (27)$$

$$|\phi_4\rangle \equiv \frac{1}{\sqrt{6}}(2|01\rangle + |10\rangle - |11\rangle)|-\rangle = |\psi_4\rangle|-\rangle. \quad (28)$$

In the above decomposition, the state is two-way separable under AB:C, but using the SHIFT invariant, we can conclude two-way separability under any bi-partition. We remark that the entanglement of the above state has recently been calculated using the geometric measure of entanglement and a generalized concurrence [41], and it is shown that the amount of entanglement is not small.

Since arbitrary single photon polarization states can be created [43] and arbitrary two-photon pure states can be created via spontaneous parametric downconversion [42], the above UPB bound entangled state can be created by mixing the four constituent two-way separable three-photon pure states. The third photon polarization appears in the four BB84 states and is implemented by simple rotations from a fixed polarization state such as  $|H\rangle$ ; see Fig. 4. To create the corresponding four two-qubit states  $|\psi_i\rangle$ , we first generate a common entangled resource state  $|\Phi\rangle$  from the downconversion source, and then operate on the signal and idler photons by local unitaries (see the Schmidt decomposition in Ref. [2] or the Appendix in Ref. [42]),

$$|\psi_1\rangle = -(U\sigma_z) \otimes (\sigma_z U)|\Phi\rangle \quad (29)$$

$$|\psi_2\rangle = (HU) \otimes (UH)|\Phi\rangle \quad (30)$$

$$|\psi_3\rangle = (\sigma_x UH) \otimes (\sigma_x U)|\Phi\rangle \quad (31)$$

$$|\psi_4\rangle = (\sigma_y U) \otimes (\sigma_y UH)|\Phi\rangle, \quad (32)$$

where  $H$  is the Hardamard gate,  $\sigma$ 's are Pauli matrices,  $|\Phi\rangle$  is the entangled resource state

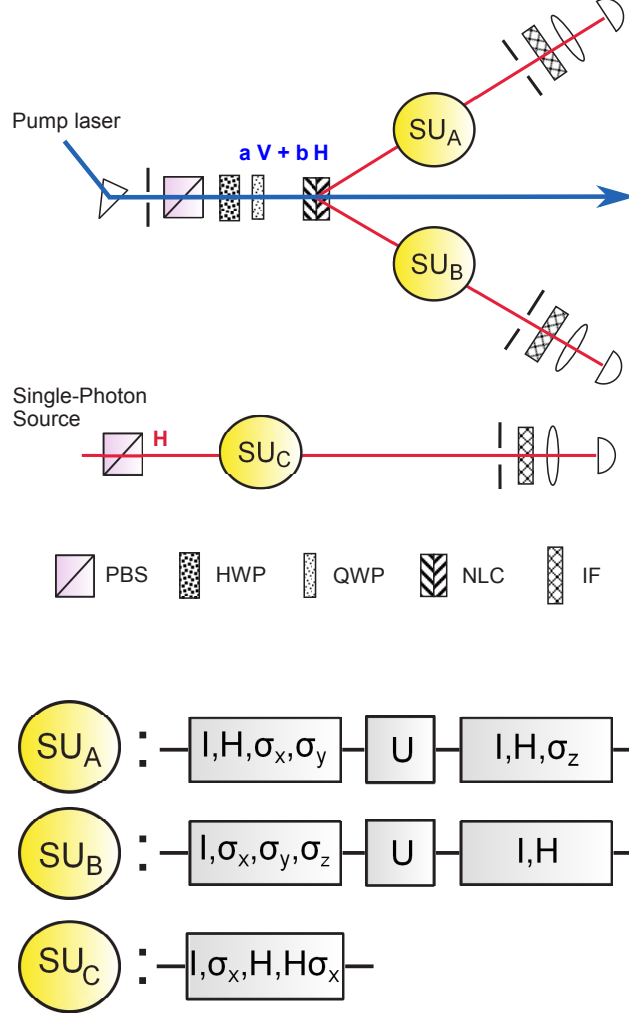


FIG. 4: (color online) Scheme for creating the three-qubit UPB bound entangled state, using a downconversion setup for generating entangled two-photon states and a single-photon source for generating the corresponding BB84 states.  $SU$ 's represent switchable unitary gates for the respective photons. The pump polarization state before the downconversion crystals is  $a|V\rangle + b|H\rangle$ , which is then downconverted to, e.g.,  $a|HH\rangle + b|VV\rangle$  (as in the type-I process of generating entangled pairs [21]). PBS represents a polarizing beam splitter, HWP represents a half-wave plate, QWP represents a quarter-wave plate, NLC represents the nonlinear crystals used for type-I downconversion, and IF represents an interference filter. For the purpose of creating the UPB bound entangled state, the choice of  $a$  and  $b$  is  $a \approx 0.934$  and  $b \approx 0.357$ ; see Eq. (33). The exact form of the unitary gate  $U$  is shown in Eq. (34). The single-photon source can also be realized by producing unentangled photon pairs via downconversion and heralded by triggering one of the photons.

and  $U$  is a single-qubit unitary gate

$$|\Phi\rangle = \sqrt{(3 + \sqrt{5})/6} |00\rangle + \sqrt{(3 - \sqrt{5})/6} |11\rangle \approx 0.934172|00\rangle + 0.356822|11\rangle \quad (33)$$

$$U = \begin{pmatrix} \frac{\sqrt{5}-1}{\sqrt{10-2\sqrt{5}}} & \sqrt{\frac{2}{5-\sqrt{5}}} \\ \sqrt{\frac{2}{5-\sqrt{5}}} & \frac{-\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}} \end{pmatrix} \approx \begin{pmatrix} 0.525731 & 0.850651 \\ 0.850651 & -0.525731 \end{pmatrix}. \quad (34)$$

The exact forms of  $U$  and  $|\Phi\rangle$  are not very illuminating and for the actual implementation the approximate forms are sufficient. All the above local unitaries can be implemented by waveplates [43]. We only need to randomly generate any of the above four states and the associated single photon states, and the statistical mixture of the outcome will be the desired bound entangled state. See Fig. 4 for the schematic setup. We remark that for the ease of implementation we have let the local unitary  $U$  be always on, but other simple gates such as  $\sigma_z$ ,  $\sigma_y$ ,  $\sigma_x$  and  $H$  need to be switched on and off depending on which state is generated, and this can be done by Pockels cells or by liquid crystals [12]. Furthermore, the overall phase factors can be ignored, as we are concerned with the mixture of the states.

## VI. CONCLUDING REMARKS

Different physical systems, such as liquid NMR, trapped ions, superconducting qubits, or photons may have their own preferred ways of implementing bound entangled states, but the techniques may also be borrowed from one another. After reviewing the schemes for implementing Smolin's state, we have proposed schemes for creating various classes of bound entangled states with photon polarization, including Acín-Bruß-Lewenstein-Sanpara states, Dür's states, Lee-Lee-Kim bound entangled states, and an unextendible-product-basis bound entangled state. These states, once existing only in theory, can now be practically realized and tested, e.g., via tomography or Bell inequalities in the laboratory. Some of them turn out to be useful in information concentration [4], bi-partite activation [5], multi-partite superactivation [6] and secure key distillation [7], as well as for providing a resource for certain zero-capacity quantum channels [9].

So far we have not discussed the issue of noise. Let us, for example, consider a quantum state that undergoes the following noisy quantum channel:  $\rho \rightarrow \rho(\epsilon) = (1 - \epsilon)\rho + \epsilon\mathbb{1}/2^N$ , namely, under a depolarizing channel. The channel only decreases entanglement content of states, and undistillable states remain undistillable. As long as it remains entangled,

the state will still be bound entangled. Indeed, all the bound entangled states discussed above have finite (nonzero) amount of entanglement, as, e.g., quantified by the geometric measure [23, 41] or negativity across certain partition, or by the construction of entanglement witnesses [26]. This means that there exists a finite range of  $\epsilon$  such that  $\rho(\epsilon)$  is still entangled [1, 44], and hence bound entangled. This is a good feature for experimental implementations in order to allow for the statistically significant observation of bound entanglement [12, 35]. Furthermore, even if there are small errors in the apparatus settings, it will in principle be possible to apply tailored noise to overcome experimental imperfections such that the resultant state is still bound entangled [12–14]. However, if too much noise is introduced, the entanglement will be washed out. The noise form that we discuss here is perhaps the simplest one. Other forms of the noise channel can in principle be treated as well.

We end by remarking that the first bound entangled states were found by Horodecki in the bi-partite systems of Hilbert spaces  $\mathcal{C}^3 \otimes \mathcal{C}^3$  and  $\mathcal{C}^2 \otimes \mathcal{C}^4$  [45]. We do not consider these in the present manuscript, because they involve non-qubit systems. However, one may consider using, e.g., two qubits to encode a three- or four-level systems, or using other degrees of freedom, such as orbital angular momentum, as considered in the hyperentanglement [46]. This is left as a possible future direction.

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